On the non-planarity of a random subgraph

Alan Frieze[*] Michael Krivelevich[†]

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Abstract

Let G be a finite graph with minimum degree r. Form a random subgraph G_p of G by taking each edge of G into G_p independently and with probability p. We prove that for any constant $\epsilon > 0$, if $p = \frac{1+\epsilon}{r}$, then G_p is non-planar with probability approaching 1 as r grows. This generalizes classical results on planarity of binomial random graphs.

1 Introduction

Planarity is a fairly classical subject in the theory of random graphs. Already Erdős and Rényi in their groundbreaking paper [2] stated (re-casting their statement in the language of binomial random graphs) that a random graph $\mathbb{G}_{n,p}$ has a sharp threshold for non-planarity at $p = 1/n$ in the following sense: if $p = c/n$ and $c < 1$ then the random graph $\mathbb{G}_{n,p}$ is with high probability (whp) planar, while for $c > 1$ $\mathbb{G}_{n,p}$ is whp non-planar. The Erdős-Rényi argument for non-planarity had a certain inaccuracy, as was pointed by Łuczak and Wierman [8], who explained how the probable non-planarity result can be obtained by other means.

The aim of this paper is to generalize this classical non-planarity result to a much wider class of probability spaces. All graphs considered in this paper are finite. For a graph $G = (V, E)$ and $0 \leq p \leq 1$ we can define the random graph $G_p = (V, E_p)$ where each $e \in E$ is independently included in E_p with probability p. When $G = K_n$, the complete graph on n vertices, G_p becomes the binomial random graph $\mathbb{G}_{n,p}$.

[*]Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh PA15213, U.S.A. Supported in part by NSF grant CCF1013110.

[†]School of Mathematical Sciences, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv 69978, Israel. E-mail: krivelev@post.tau.ac.il. Research supported in part by a USA-Israel BSF Grant and by a grant from Israel Science Foundation.

Here is the main result of the present paper.

Theorem 1 *Let G be a finite graph with minimum degree r and let $p = \frac{1+\epsilon}{r}$, where $\epsilon > 0$ is an arbitrary constant. Then*

$$\mathbb{P}(G_p \text{ is planar}) \leq \theta_r$$

where $\lim_{r\to\infty} \theta_r = 0$.

2 Proof of Theorem 1

Our proof rests in large part on the following simple consequence of Euler's formula.

Lemma 1 *Let $G = (V, E)$ be a planar graph with n vertices and m edges and girth g. Then*

$$m \leq \frac{g(n-2)}{g-2} < n + \frac{2}{g-2}n.$$

Proof

Let f be the number of faces of a planar embedding of G. Then we have

$$m = n + f - 2 \text{ and } 2m \geq gf.$$

□

Before proving Theorem 1 it will be instructive to prove it for the special case where $G = K_n$ i.e. to show that if $p = \frac{c}{n}$ where $c > 1$ is a constant then $\mathbb{G}_{n,p}$ is non-planar whp.

2.1 $\mathbb{G}_{n,p}$

The non-planarity of $\mathbb{G}_{n,p}$ is already known even for $c = 1 + \omega n^{-1/3}$ provided $\omega \to \infty$ with n, see Łuczak, Pittel and Wierman [9], see also [10] for very accurate results on the probability of planarity in the critical window $p = (1 + O(n^{-1/3})/n$. The analysis for $c = 1 + o(1)$ is quite challenging, but for constant $c > 1$ it follows simply from some well known facts. Let G_1 be the largest connected component of $\mathbb{G}_{n,p}$ (well known to be whp the unique component of linear size for $c > 1$, the so called giant component). It is known, see e.g. [1], that whp

$$|V(G_1)| \sim xn \text{ and } |E(G_1)| \sim cn(2x - x^2)/2$$

where x is the unique $(0,1)$ solution to $x = 1 - e^{-cx}$. This gives

$$c = 1 + \frac{x}{2} + \frac{x^2}{3} + \cdots$$

and so if $c = 1 + \epsilon$, $\epsilon > 0$ and small, then $x = 2\epsilon + O(\epsilon^2)$.

Thus in this case, whp,

$$\frac{|E(G_1)|}{|V(G_1)|} \sim \frac{2}{2-x} == 1 + \epsilon + O(\epsilon^2).$$

Next let $g_0 = 10/\epsilon$. Then if X denotes the number of cycles in $\mathbb{G}_{n,p}$ of length at most g_0,

$$\mathbb{E}(X) \leq \sum_{k=3}^{g_0} n^k p^k \leq g_0 c^{g_0}.$$

So, whp, there are fewer than $\ln n$ cycles of length at most g_0. So, by removing at most $\ln n$ edges from $\mathbb{G}_{n,p}$ we obtain a sub-graph G_1' with girth higher than g_0. Now

$$\frac{|E(G_1')|}{|V(G_1')|} \sim \frac{|E(G_1)|}{|V(G_1)|} \sim 1 + \epsilon + O(\epsilon^2) > 1 + \frac{2}{g_0 - 2}$$

for small enough ϵ. Lemma 1 implies that G_1' and hence G_1 are both non-planar.

2.2 Proof of Theorem 1

All asymptotic quantities are to be interpreted for $r \to \infty$ i.e. if we say $\xi = \xi(r) = o(1)$ then we mean that $\limsup_{r\to\infty} |\xi| = 0$. This includes the notion of high probability. I.e. if an event $\mathcal{E}$ occurs with probability $1 - \xi(r)$ where $\limsup_{r\to\infty} |\xi| = 0$ then we say that $\mathcal{E}$ ocurs whp.

Notation: If X is a set of edges and A, B are disjoint sets of vertices, then $E_X(A, B)$ is the set of edges in X with one endpoint in A and one endpoint in B. Furthermore, $E_X(A)$ is the set of edges in X with both endpoints in A. We let $e_X(A, B) = |E_X(A, B)|$ and $e_X(A) = |E_X(A)|$.

Our strategy for proving Theorem 1 will be to prove the existence, whp, of a sub-graph which has large girth and sufficient edge density to apply Lemma 1. For this we will need the following lemma:

Lemma 2 *Let $0 < c_1, c_2 < 1$ be constants. Let $T = (V, E)$ be a tree on n vertices with maximum degree $\Delta = r^{o(1)}$. Let $F \subseteq \binom{V}{2}$ with $|F| = c_1 nr$. Form a random subset F_p of F by choosing every edge of F to belong to F_p independently and with probability $p = \frac{c_2}{r}$. Then the graph $G = T \cup F_p$ is non-planar, with probability $1 - O(\log r/r)$.*

Proof Set

$$\alpha = \frac{c_1 c_2}{72} \text{ and } A = \frac{10^{10}}{c_1^5 c_2^6}\,.$$

Let

$$V_0 = \{v : d_F(v) \geq Ar\}\,,$$

where $d_X(v)$ is the degree of vertex v in the graph induced by $X \subseteq F$.

Clearly

$$|V_0| \leq \frac{2c_1 n}{A}.$$

Let F' be the set of edges from F with at least one endpoint in V_0.

Case 1: $|F'| \geq c_1 nr/2$.
If $e_F(V_0) \geq c_1 nr/4$, then the Chernoff bound for the binomial distribution implies that with probability $1 - e^{-\Omega(n)}$ we have

$$e_{F_p}(V_0) \geq \frac{c_1 nr}{4} \cdot \frac{c_2}{r}(1 - o(1)) > \frac{c_1 c_2 n}{5} \geq \frac{c_2 A}{10}|V_0| > 4|V_0|.$$

In which case, the subgraph induced by $E_{F_p}(V_0)$ forms a non-planar graph. Hence we can assume from now on that F' has at least $c_1 nr/4$ edges with at most one endpoint in V_0.

Define

$$U_0 = \left\{v \notin V_0 : d_{F'}(v) \geq \frac{c_1 r}{8}\right\}\,.$$

Then

$$e_F(U_0, V_0) \geq |F'| - e_F(V_0) - \frac{c_1 nr}{8} \geq \frac{c_1 nr}{8}.$$

Now the Chernoff bounds imply that with probability $1 - e^{-\Omega(n)}$ we have

$$e_{F_p}(U_0, V_0) \geq \frac{c_1 nr}{8} \cdot \frac{c_2}{r}(1 - o(1)) \geq \frac{c_1 c_2 n}{9}.$$

So if $|U_0| \leq \alpha n$ then

$$\frac{e_{F_p}(U_0, V_0)}{|U_0| + |V_0|} \geq \frac{c_1 c_2 n/9}{c_1 c_2 n/72 + 2c_1 n/A} \geq 4.$$

In which case, the subgraph induced by $E_{F_p}(V_0)$ forms a non-planar graph.

If $|U_0| \geq \alpha n$, then define a (random) subset W_0 by:

$$W_0 = \{v \in U_0 : d_{F_p}(v, V_0) \geq 5\}\,.$$

The distribution of $|W_0|$ dominates $Bin(|U_0|, q_1)$ where

$$q_1 = \mathbb{P}(Bin(c_1 r/8, c_2/r) \geq 5) \geq \binom{\frac{c_1 r}{8}}{5}\left(\frac{c_2}{r}\right)^5\left(1 - \frac{c_1}{r}\right)^{c_1 r/8} \geq (1 - o(1))\left(\frac{c_1 c_2}{8}\right)^5 \cdot \frac{e^{-c_1^2/8}}{120}.$$

The Chernoff bounds then imply that with probability $1 - e^{-\Omega(n)}$ we have

$$|W_0| \geq \frac{|U_0|q_1}{2} \geq \frac{\alpha n q_1}{2},$$

and by definition $e_{F_p}(W_0, V_0) \geq 5|W_0|$. This is at least $4|W_0 \cup V_0|$. In which case, the subgraph induced by $E_{F_p}(V_0, W_0)$ forms a non-planar graph. This completes the analysis for Case 1.

Case 2: $|F'| \leq c_1 nr/2$.
Define $F'' = F \setminus F'$ and observe that $|F''| \geq c_1 nr/2$, and by definition the maximum degree of F'' is at most Ar.

Observe that $T \cup F''_p$ has with probability $1 - e^{-\Omega(n)}$ at least

$$n - 1 + \frac{c_1 nr}{2} \cdot \frac{c_2}{r}(1 - o(1)) > n(1 + \epsilon)$$

edges for some positive $\epsilon = \epsilon(c_1, c_2)$. It thus suffices to show that the number of "short" cycles in $T \cup F''_p$ is $o(n)$ whp, and then use Lemma 1.

For constants $\ell, t = O(1)$ let us estimate the expected number of cycles of length ℓ in $T \cup F''_p$ having t edges from T. We choose an initial vertex v in n ways, then decide about the placement of the edges of T in the cycle in $O(1)$ ways. We thus get a sequence $P_1 * P_2 \cdots * P_{t+1}$, where the stars correspond to edges from T and P_i is a path of length ℓ_i for $i = 1, 2, \ldots, t+1$. Now, a path of length ℓ_i in F'', starting from a given point, can be chosen in at most $(Ar)^{\ell_i}$ ways, an edge from T from a given vertex can be chosen in at most $\Delta(T) = r^{o(1)}$ ways, and finally the last path of length ℓ_{t+1}, connecting two already chosen vertices, can be chosen in at most $(Ar)^{\ell_{t+1}-1}$ ways. Altogether the number of such cycles in $T \cup F''$ is $n \cdot r^{o(1)} \cdot O\left(r^{\ell_1 + \ldots + \ell_{t+1} - 1}\right)$ ways. The probability for such a cycle to survive in $T \cup F''_p$ is $O\left(r^{-(\ell_1 + \ldots + \ell_{t+1})}\right)$. We thus expect $O(n/r)$ such cycles. Summing over all choices of ℓ and t we get that the expected number of cycles of length $O(1)$ in $T \cup F''_p$ is $O(n/r)$, and thus the Markov inequality implies we get fewer than $n/\log r$ cycles, with probability $1 - O(\log r/r)$. By choosing ℓ suffcently large and deleting one edge from each cycle of length at most ℓ, we get a graph of large constant girth, with n vertices and at least $(1 + \epsilon/2)n$ edges — which is non-planar, by Lemma 1. $\square$

We now set about using the above lemma. We let $G_p = G_1 \cup G_2$ where $G_i = G_{p_i}, i = 1, 2$, and $p_1 = \frac{1+\epsilon/2}{r}$ and $(1 - p_1)(1 - p_2) = 1 - p$ so that $p_2 = \frac{\epsilon + O(\epsilon^2)}{2r}$.

2.3 Initial Tree Growth

We begin by repeatedly choosing a vertex $v \in V$ and analysing a restricted breadth search (RBFS) from v until we succeed in obtaining a certain condition, see (1) below. Basically, we need to find v which has sufficiently many neighbors in G_1. So, let $S_0 = \{v\}$. In general

let

$$S_{i+1} = \bigcup_{w \in S_i} (RN(w) \setminus (T_i \cup B))$$

where

- $B, |B| = o(r)$, is a set of vertices that already been rejected by our search.
- $T_i = \bigcup_{j=0}^{i} S_j$.
- $RN(w)$ denotes the first $B_1 = Bin(r_1, p_1)$, $r_1 = r - O(i_0 \log r)$, neighbors of w in G_1, where
$$i_0 = \log^3 r,$$
By first we assume that $V(G) = [n]$ for some integer n. Then we mean that we try the first r_1 G-neighbors of a vertex w in numerical value to see if they are neighbors of w in G_1. The edges found will be part of a subgraph H_1 and we only keep the first edge found to each vertex added. In this way, H_1 will be a tree.

Our initial aim in RBFS is to find a smallest k such that

$$i_0 \leq |T_k| \leq 2i_0 \text{ and } \frac{|S_k|}{|T_k|} \in \left[\frac{\epsilon}{4}, \frac{3\epsilon}{4}\right]. \tag{1}$$

Let $d = \log^{1/2} r$. We first look for v such that $|S_1| \geq d$. This is quite simple. Let $l_0 = (2d)^d = o(r)$ and suppose that we have already examined $v_1, v_2, \ldots, v_l$, $l \leq l_0$, without success. We choose $v \notin B_l = \{v_1, v_2, \ldots, v_l\}$ and examine the first $r - o(r)$ neighbours of v that are not in B_l. The probability that v has at least d neighbors in G_1 is greater than $\binom{r-o(r)}{d} p_1^d (1-p_1)^{r-o(r)} \geq d^{-d}$. So, the probability we have not found v with large enough degree after l_0 trials is less than $(1 - d^{-d})^{l_0} = o(1)$. Furthermore, the probability v has more than $i_0/2$ neighbors is less than $\binom{r}{i_0/2} p_1^{i_0/2} \leq e^{-i_0}$. We can therefore assume that we can find a suitable v with $|S_1| \geq d$ where $B = B_l$ is of size $o(r)$.

Suppose now that $S_i, T_i, i \geq 1$ do not satisfy (1) and that $|T_i| \leq 2i_0$. We observe first that the distribution of the size of S_{i+1} is dominated by $Bin((r - o(r))|S_i|, p_1)$. In fact we bound $|S_{i+1}|$ from above by the number of edges from S_i to S_{i+1}. We examine the first $r - o(r)$ G-neighbors of each v in S_i and include an edge vw in our count if the edge vw is in G_1. Therefore

$$\mathbb{P}(|S_{i+1}| \geq (1 + 2\epsilon/3)s \mid |S_i| = s) \leq e^{-\Omega(\epsilon^2 s)}. \tag{2}$$

We can also argue that $|S_{i+1}|$ dominates a binomial $Bin(|S_i|(r - o(r)), p_1)$. The $o(r)$ term here differs from the one used in the upper bound. We will have to exclude edges to those G-neighbors that have already been placed in S_{i+1} and to those G-neighbors in B_l. Because we are looking for a lower bound which is less than i_0, we can claim to get at least the result of $|S_i|(r - o(r))$ trials with success probability p_1. Therefore

$$\mathbb{P}(|S_{i+1}| \leq (1 + \epsilon/3)s \mid |S_i| = s) \leq e^{-\Omega(\epsilon^2 s)}. \tag{3}$$

So we can assume that $\alpha_1 \geq d$ and $|S_i|/|S_{i-1}| = \alpha_i \in [(1+\epsilon/3), (1+2\epsilon/3)]$ for $i \geq 2$. And then

$$\frac{|S_i|}{|T_i|} = \frac{\alpha_1\alpha_2\cdots\alpha_i}{1+\alpha_1+\alpha_1\alpha_2+\cdots+\alpha_1\alpha_2\cdots\alpha_i}. \tag{4}$$

The expression (4) is minimised (resp. maximised) by putting $\alpha_i = (1+\epsilon/3)$ (resp. $= (1+2\epsilon/3)$) for $i \geq 2$. It follows that whp

$$\frac{|S_i|}{|T_i|} = \frac{\alpha_1\theta^{i-1}(\theta-1)}{1+\alpha_1(\theta^i-1)}$$

for some $\theta \in [(1+\epsilon/3), (1+2\epsilon/3)]$. Thus we will achieve (1) whp. Here we use the fact that the sum of the failure probabilities in (2),(3) is bounded by $\sum_{s\geq d} e^{-\Omega(\epsilon^2 s)} = o(1)$.

2.4 Remaining Tree Growth

Let us consider the current tree T_k, which is of size $\Omega(i_0)$, and its frontier S_k of size $|S_k| = s_k = \Theta(\epsilon|T_k|)$. Choose $r_1 = r - o(r)$ arbitrary edges incident to each vertex of S_k, denote the obtained set by E_k, $|E_k| \leq rs_k$. If E_k has $\Theta(rs_k)$ edges inside $V(T_k)$, then sprinkling the edges of E_k with probability p_2 produces whp a non-planar graph on $V(T_k)$ by Lemma 2. We can therefore assume that E_k has at least $(1-\frac{\epsilon}{10})rs_k$ edges between S_k and $V \setminus T_k$.

Let $V_0 = \{v \notin T_k : d_{E_k}(v, S_k) \geq r\ln r\}$. Clearly, $|V_0| \leq s_k/\ln r$. If E_k has at least $\epsilon rs_k/10$ edges between S_k and V_0, then in the random subset of E_k, formed by taking each edge independently and with probability p_1, there is whp a set W_0 of $|W_0| = \Theta(s_k)$ vertices $v \in S_k$, whose degrees η_v into V_0 are at least three. Indeed, there will be at least $\epsilon s_k/20$ vertices S'_k in S_k that have at least $\epsilon r/20$ S_k neighbours in V_0. Each vertex in S'_k has a probability of at least $\xi = \binom{\epsilon r/20}{3}p_1^3(1-p_1)^{\epsilon r/20-3} \geq \epsilon^5 10^{-10}$ of having $\eta_v \geq 3$, and these events are independent. Thus whp $|W_0| \geq |S'_k|\xi/2$ and the bipartite subgraph of G_p induced by W_0, V_0 has more than $2(|W_0|+|V_0|)$ edges and so is non-planar. We can assume therefore that E_k has at least $(1-\frac{\epsilon}{5})rs_k$ edges between S_k and $V_1 = V \setminus (T_k \cup V_0)$. Denote this set of edges by F_k.

Form a random subgraph R_k of F_k by taking each edge independently and with probability p_1.

P1 Then the Chernoff bounds imply that with probability $1 - e^{-\Omega(s_k)}$, $|R_k| \geq (1+\frac{\epsilon}{5})s_k$.

P2 Furthermore, we will show next that with probability $1-\epsilon_1(r)$ at most $2s_k/\log r$ of these edges are incident with vertices in $V_2 \subseteq S_k$ whose degree in R_k is more than $\log\log r$.

The value of $\epsilon_1(r) = \epsilon_1'(r) + \epsilon_1''(r)$ is obtained from (5) and (6) below. Indeed, if $v \in S_k$ then

$$\mathbb{P}(d_{R_k}(v) \geq \log\log r) \leq \mathbb{P}(Bin(r, p_1) \geq \log\log r) \leq q_2 = \left(\frac{2e}{\log\log r}\right)^{\log\log r}$$

and

$$\mathbb{P}(d_{R_k}(v) \geq \log r) \leq \mathbb{P}(Bin(r, p_1) \geq \log r) \leq q_3 = \left(\frac{2e}{\log r}\right)^{\log r}.$$

Thus the number of edges in R_k that are incident with $v \in V_2$ is bounded by $Bin(s_k, q_2)\log r + Bin(s_k, q_3)r$. We observe that because $s_k \geq \epsilon i_0/8 \gg \log^2 r$ we can write

$$\mathbb{P}(Bin(s_k, q_2) \geq s_k/\log^2 r) \leq \epsilon_1'(r) = (e\log^2 r\, q_2)^{s_k/\log^2 r} \tag{5}$$

and

$$\mathbb{P}(Bin(s_k, q_3) \geq s_k/r^2) \leq \epsilon_1''(r) = \begin{cases} r^3 q_3 & 0 \leq s_k < r^3 \\ (er^2\, q_3)^{s_k/r^2} & s_k \geq r^3 \end{cases}. \tag{6}$$

Let N_k be the set of neighbors of S_k defined by edges in R_k. We observe that $|N_k|$ is the sum of independent Bernouilli random variables. We consider two cases depending on the value of $\mathbb{E}(|N_k|)$ w.r.t. the random set R_k. Splitting the argument this way will not condition R_k or N_k.

Case 1: $\mathbb{E}(|N_k|) \geq \left(1 + \frac{\epsilon}{10}\right) s_k$.

We first observe that

$$\mathbb{P}\left(|N_k| \leq \left(1 + \frac{\epsilon}{20}\right) s_k\right) \leq e^{-\epsilon^2 s_k/1000}. \tag{7}$$

We therefore assume that

$$|N_k| \geq \left(1 + \frac{\epsilon}{20}\right) s_k.$$

R_k contains a subset R_k' of size $\nu_k = \left(1 + \frac{\epsilon}{25}\right) s_k$ such that the degrees of all the vertices in S_k w.r.t. R_k' are at most $\log\log r$, and every vertex outside T_k is incident to at most one edge from R_k' and there are ν_k vertices outside T_k incident to an edge in R_k'. We obtain this by removing edges incident with V_2 and by then deleting edges incident with N_k to get degree at most one. Use R_k' to form the next frontier of size $(1 + \frac{\epsilon}{25})s_k$, composed of the endpoints of the edges of R_k' outside S_k; proceed to the next round.

Case 2: $\mathbb{E}(|N_k|) \leq \left(1 + \frac{\epsilon}{10}\right) s_k$.

Q1 $|N_k| \leq (1 + \frac{\epsilon}{8})s_k$ whp. Indeed,

$$\mathbb{P}\left(|N_k| \geq \left(1 + \frac{\epsilon}{8}\right) s_k\right) \leq e^{-\epsilon^2 s_k/5000}.$$

Q2 $|R_k| \geq (1+\frac{\epsilon}{5})s_k$ whp. Indeed, $|R_k| = Bin(|F_k|, p_1)$ and so

$$\mathbb{P}\left(|R_k| \leq \left(1+\frac{\epsilon}{5}\right) s_k\right) \leq e^{-\epsilon^2 s_k/1200}$$

for small $\epsilon > 0$.

Q3 There are $o(s_k)$ short cycles in $T_k \cup R_k$ whp. For this calculation we consider the graph Γ_k induced by the edges in $E(T_k) \cup F_k$. This has vertex set $V(T_k) \cup N_k$. Here the expectation calculation is quite similar to that of the lemma. We use the fact that V_0 has been excluded, and therefore all relevant vertices outside of T_k have their degrees into S_k bounded by $r \ln r$. Also all degrees in T_k are $r^{o(1)}$ by our construction.
Details: For constants $\ell, t = O(1)$ let us estimate the expected number of cycles of length ℓ in Γ_k having t edges from T_k. We choose an initial vertex v in $O(s_k)$ ways, then decide about the placement of the edges of T_k in the cycle in $O(1)$ ways. We thus get a sequence $P_1 * P_2 \cdots * P_{t+1}$, where the stars correspond to edges from T_k and P_i is a path of length ℓ_i for $i = 1, 2, \ldots, t+1$ using edges in F_k. Now, a path of length ℓ_i using edges in F_k, starting from a given point, can be chosen in at most $(r \log r)^{\ell_i}$ ways, an edge from T_k from a given vertex can be chosen in at most $\Delta(T_k) = r^{o(1)}$ ways, and finally the last path of length ℓ_{t+1}, connecting two already chosen vertices, can be chosen in at most $(r \log r)^{\ell_{t+1}-1}$ ways. Altogether the number of such cycles in Γ is $s_k \cdot r^{o(1)} \cdot \tilde{O}\left(r^{\ell_1+\ldots+\ell_{t+1}-1}\right)$ ways. The probability for such a cycle to survive in $T_k \cup R_k$ is $O\left(r^{-(\ell_1+\ldots+\ell_{t+1})}\right)$. We thus expect $O(s_k/r^{1-o(1)})$ such cycles. Summing over all choices of ℓ and t we get that the expected number of cycles of length $O(1)$ in $T \cup F_p''$ is $O(s_k/r)$, and thus the Markov inequality implies we get fewer than $s_k/\log r$ cycles, with probability $1 - \tilde{O}(1/r)$.

By choosing ℓ sufficiently large and removing edges from the short cycles (length $\leq \ell$) leaves a graph of average degree $2 + \Theta(\epsilon)$ and without short cycles. This is non-planar by Lemma 1.

As a final note in proof, we argue about the probability that this construction fails. We have seen that the initial tree growth in Section 2.3 succeeds whp. The success of the remaining tree growth rests on the probabilities in **P1,P2** being high enough. These events need to happen multiple times, whereas other events are only required to occur once.

For **P1** and (7) we verify that $\sum_{t \geq i_0} e^{-\Omega(t)} = o(1)$ and for **P2** we verify that

$$\sum_{t \geq i_0} (e \log^2 r\, q_2)^{-t/\log^2 r} + \sum_{t=0}^{r^3} r^3 q_3 + \sum_{t \geq r^3} (er^2\, q_3)^{t/r^2} = o(1).$$

3 Concluding remarks

We have proven that for every finite graph G of minimum degree $r \gg 1$, a random subgraph G_p of G, with $p = p(r) = \frac{1+\epsilon}{r}$ and $\epsilon > 0$ being an arbitrary small constant, is whp non-planar. This generalizes the classical non-planarity results for binomial random graphs $\mathbb{G}_{n,p}$. It should be noted that for a statement of such generality we cannot hope to have a matching lower bound on $p(r)$. Indeed, if G is a collection of, say, 2^{r^3} vertex disjoint cliques K_{r+1}, then for any constant $c > 0$, the random subgraph G_p, $p = c/n$, retains whp one of the cliques K_{r+1} in full and is thus whp non-planar.

Notice that our proof shows in fact that under the conditions of Theorem 1 the random subgraph G_p is typically non only non-planar, but has an unbounded genus. This can be obtained by substituting Euler's formula with its analog for surfaces of bounded genus.

It would be interesting to determine whether under the same conditions the random graph G_p has whp a minor of a complete graph K_t for $t = t(r)$ growing with r. For the case of binomial random graphs $\mathbb{G}_{n,p}$ this is indeed the case, in particular, Fountoulakis, Kühn and Osthus showed [3] that for any $c > 1$, the random graph $\mathbb{G}_{n,p}$ with $p = c/n$ has whp a complete minor of order $\sqrt{n}$. (See also [6] for results for other values of $p = p(n)$, and [4] for results on random regular graphs and for $\mathbb{G}_{n,p}$ in the slightly supercritical regime). While for the case of $\mathbb{G}_{n,p}$ proving the probable existence of a complete minor of order $\tilde{\Theta}(\sqrt{n})$ for $p = c/n$, $c > 1$, is in fact not so complicated, it is unclear whether the methods of this paper are directly applicable to show a similar result for the case of a random subgraph of a generic graph G.

The main theorem of this paper can be viewed as yet another contribution to a growing sequence of results about properties of random subgraphs of graphs of given maximum degree. We can mention here [7], who showed that if G is a finite graph of minimum degree r and $p = \frac{1+\epsilon}{r}$, then the random graph G_p contains whp a path of length linear in r, and also [5], where it is proven that under the same assumptions on the base graph G and when taking $p = \frac{(1+o(1))\ln r}{r}$, the random graph G_p contains whp a path of length at least r, in both cases substantially generalizing classical results about binomial random graphs. One can certainly anticipate more results of this type to appear in the near future.

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